REGULAR SEMISIMPLE HESSENBERG VARIETIES WITH COHOMOLOGY RINGS GENERATED IN DEGREE TWO

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Abstract. A regular semisimple Hessenberg variety $\mathrm{Hess}(S,h)$ is a smooth subvariety of the flag variety determined by a square matrix S with distinct eigenvalues and a Hessenberg function h. The cohomology ring $H^*(\mathrm{Hess}(S,h))$ is independent of the choice of S and is not explicitly described except for a few cases. In this paper, we characterize the Hessenberg function h such that $H^*(\mathrm{Hess}(S,h))$ is generated in degree two as a ring. It turns out that such h is what is called a (double) lollipop.

1. Introduction

The flag variety $\mathrm{Fl}(n)$ consists of nested sequences of linear subspaces in the complex vector space $\mathbb{C}^n$:

$$\mathrm{Fl}(n) = \{V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = i \quad (\forall i \in [n] = \{1,2,\dots,n\})\}.$$

A Hessenberg function $h\colon [n] \to [n]$ is a monotonically non-decreasing function satisfying $h(j) \geq j$ for any $j \in [n]$. We often express a Hessenberg function h as a vector $(h(1),\dots,h(n))$ by listing the values of h. Given an $n \times n$ matrix A and a Hessenberg function h, a Hessenberg variety $\mathrm{Hess}(A,h)$ is defined as

$$\mathrm{Hess}(A,h) = \{V_\bullet \in \mathrm{Fl}(n) \mid AV_i \subset V_{h(i)} \quad (\forall i \in [n])\}$$

where the matrix A is regarded as a linear operator on $\mathbb{C}^n$. Note that $\mathrm{Hess}(A,h) = \mathrm{Fl}(n)$ if $h = (n,\dots,n)$.

The family of Hessenberg varieties $\mathrm{Hess}(A,h)$ contains important varieties such as Springer fibers (A is nilpotent and $h = (1,2,\dots,n)$), Peterson varieties (A is regular nilpotent and $h = (2,3,\dots,n,n)$), and permutohedral varieties (A is regular semisimple and $h = (2,3,\dots,n,n)$), which are toric varieties with permutohedra as moment polytopes.

Among $n \times n$ matrices, regular semisimple ones S (i.e. matrices S having distinct eigenvalues) are generic and $\mathrm{Hess}(S,h)$ is called a regular semisimple Hessenberg variety. The regular semisimple Hessenberg variety $\mathrm{Hess}(S,h)$ has nice properties. For instance, it is smooth and its cohomology $H^*(\mathrm{Hess}(S,h))$ becomes a module over the symmetric group $\mathfrak{S}_n$ on $[n]$ by Tymoczko's dot action [20]. Remarkably, the solution of Shareshian–Wachs conjecture [18] by Brosnan and Chow [5] (and Guay-Paquet [10]) connected $H^*(\mathrm{Hess}(S,h))$ as an $\mathfrak{S}_n$-module and chromatic symmetric functions on certain graphs. This opened a way to prove the famous Stanley–Stembridge conjecture in graph theory through the geometry or topology of Hessenberg varieties and motivated us to study $H^*(\mathrm{Hess}(S,h))$. Note that $H^*(\mathrm{Hess}(S,h))$ (indeed the diffeomorphism type of $\mathrm{Hess}(S,h)$) is independent of the choice of S. We write the regular semisimple Hessenberg variety $\mathrm{Hess}(S,h)$ as $X(h)$ for brevity since our concern in this paper is its cohomology ring.

The $\mathfrak{S}_n$-module structure on $H^*(X(h))$ is determined in some cases (e.g. [12]). In particular, that on $H^2(X(h))$ was explicitly described by Chow [7] combinatorially (through the theorem by Brosnan-Chow mentioned above) and by Cho-Hong-Lee [6] geometrically. Motivated by their works, Ayzenberg and the authors [4] reproved their results by giving explicit additive generators of $H^2(X(h))$ in terms of GKM theory.

The ring structure on $H^*(X(h))$ is not explicitly described except for a few cases. Remember that $X(h)$ for $h = (n,\dots,n)$ is the flag variety $\mathrm{Fl}(n)$ and $H^*(\mathrm{Fl}(n))$ is generated in degree 2 as a ring. Moreover, $X(h)$ for $h = (2,3,\dots,n,n)$ is the permutohedral variety and $H^*(X(h))$ is also generated in degree 2 as a ring. On the other hand, for $h = (h(1),n,\dots,n)$ with $h(1)$ arbitrary, a result of [2] shows that $H^*(X(h))$ is generated

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in degree 2 as a ring if and only if $h(1) = 2$ or n, where $X(h) = \mathrm{Fl}(n)$ for the latter case. Therefore, it is natural to ask when $H^*(X(h))$ is generated in degree 2 as a ring. The answer is the following, which is our main result in this paper.

Theorem 1.1. *Assume that $h(j) \geq j+1$ for $j \in [n-1]$. Then $H^*(X(h))$ is generated in degree 2 as a ring if and only if h is of the following form (1.1) for some $1 \leq a < b \leq n$,*

$$h(j) = \begin{cases} a+1 & (1 \leq j \leq a) \\ j+1 & (a < j < b) \\ n & (b \leq j \leq n). \end{cases} \tag{1.1}$$

Remark 1.1. (1) $X(h)$ is connected if and only if $h(j) \geq j+1$ for any $j \in [n-1]$. When $X(h)$ is not connected, each connected component of $X(h)$ is a product of smaller regular semisimple Hessenberg varieties.

(2) $X(h)$ is the flag variety $\mathrm{Fl}(n)$ when $(a,b) = (n-1,n)$ and is the permutohedral variety when $(a,b) = (1,n)$.

(3) We will give an explicit presentation of the ring structure on $H^*(X(h))$ for h of the form (1.1) in a forthcoming paper [17].

We can visualize a Hessenberg function h by drawing a configuration of the shaded boxes on a square grid of size $n \times n$, which consists of boxes in the i-th row and the j-th column satisfying $i \leq h(j)$. Since $h(j) \geq j$ for any $j \in [n]$, the essential part is the shaded boxes below the diagonal. For example, Figure 1 below is the configurations of two Hessenberg functions h of the form (1.1) with $n = 11$: one is $h = (2,3,4,5,6,7,11,11,11,11)$ where $(a,b) = (1,7)$ and the other is $h = (4,4,4,5,6,7,11,11,11,11)$ where $(a,b) = (3,7)$. We often identify a Hessenberg function h with its configuration.

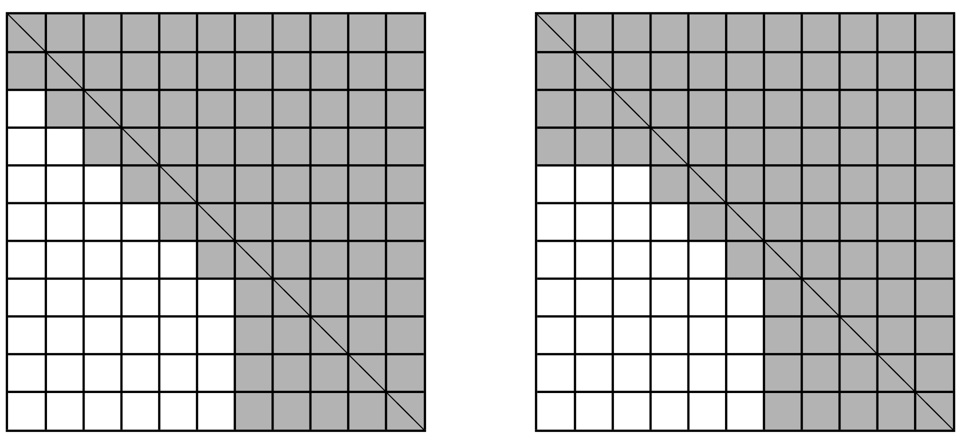

Figure 1. The configurations for $h = (2,3,4,5,6,7,11,11,11,11)$ and $h = (4,4,4,5,6,7,11,11,11,11)$

The chromatic symmetric functions and LLT polynomials associated with h of the form (1.1) are studied from the viewpoint of combinatorics in [8, 13], and when $a = 1$ or $b = n$, the corresponding Hessenberg functions

$$h = (2,3,\dots,b,n,\dots,n) \quad \text{or} \quad (a+1,\dots,a+1,a+2,\dots,n-1,n,n)$$

are called lollipops in those papers, so the Hessenberg function of the form (1.1) may be called a double lollipop.

The paper is organized as follows. In Section 2, we review GKM theory to compute the equivariant cohomology of $X(h)$. We prove the "only if" part in Theorem 1.1 in Section 3. Indeed, we consider a Morse-Bott function f_h on $X(h)$, where the inverse image of the minimum or maximum value of f_h is a regular semisimple Hessenberg variety $X(h')$ with h' of size one less than that of h. Then a property of the

Morse-Bott function f_h shows the surjectivity of the restriction map $H^*(X(h);\mathbb{Q}) \to H^*(X(h');\mathbb{Q})$, and this enables us to use an inductive argument to prove the "only if" part. In Section 4, we prove the "if" part in Theorem 1.1 by showing that $X(h)$ is the total space of a fiber bundle over a compact smooth toric variety with a product of flag varieties as the fiber when h is of the form (1.1). Since the cohomology rings of the base space and the fiber are both generated in degree two, so is the cohomology ring of $X(h)$.

2. Regular semisimple Hessenberg varieties

We first recall some properties of a regular semisimple Hessenberg variety $X(h)$.

Theorem 2.1 ([16])**.**

(1) *$X(h)$ is smooth.*
(2) *$\dim_{\mathbb{C}} X(h) = \sum_{j=1}^{n}(h(j)-j)$.*
(3) *$X(h)$ is connected if and only if $h(j) \geq j+1$ for $\forall j \in [n-1]$.*
(4) *$H^{odd}(X(h)) = 0$ and the $2k$-th Betti number of $X(h)$ is given by*

$$\#\{w \in \mathfrak{S}_n \mid \ell_h(w) = k\}$$

where

$$\ell_h(w) = \#\{1 \leq j < i \leq n \mid w(j) > w(i),\ i \leq h(j)\}. \tag{2.1}$$

For calculation of the cohomology ring of $X(h)$, we use equivariant cohomology which we shall explain. We assume that the matrix S in $X(h) = \mathrm{Hess}(S,h)$ is a diagonal matrix. Let T be an algebraic torus consisting of diagonal matrices in the general linear group $\mathrm{GL}_n(\mathbb{C})$. The linear action of T on $\mathbb{C}^n$ induces an action on the flag variety $\mathrm{Fl}(n)$ and preserves $X(h)$ since S commutes with T. The fixed point sets of the T-actions on $X(h)$ and $\mathrm{Fl}(n)$ consist of all permutation flags, that is,

$$X(h)^T = \mathrm{Fl}(n)^T \cong \mathfrak{S}_n. \tag{2.2}$$

Since T can naturally be identified with $(\mathbb{C}^*)^n$, the classifying space BT of T is $B(\mathbb{C}^*)^n = (\mathbb{C}P^\infty)^n$. Let $p_i\colon T \to \mathbb{C}^*$ be the projection on the i-th diagonal component of T and $t_i = p_i^*(t) \in H^2(BT)$ where $p_i^*\colon H^*(B\mathbb{C}^*) \to H^*(BT)$ and $t \in H^2(B\mathbb{C}^*)$ is the first Chern class of the tautological line bundle over $B\mathbb{C}^* = \mathbb{C}P^\infty$. Then

$$H^*(BT) = \mathbb{Z}[t_1,\dots,t_n]. \tag{2.3}$$

The equivariant cohomology of the T-variety $X(h)$ is defined as

$$H_T^*(X(h)) := H^*(ET \times_T X(h))$$

where ET is the total space of the universal principal T-bundle $ET \to BT$ and $ET \times_T X(h)$ is the orbit space of the product $ET \times X(h)$ by the diagonal T-action. The projection $ET \times X(h) \to ET$ on the first factor induces a fibration

$$X(h) \xrightarrow{\rho} ET \times_T X(h) \xrightarrow{\pi} BT.$$

Since $H^{odd}(X(h)) = 0$ as in Theorem 2.1 and $H^{odd}(BT) = 0$, the Serre spectral sequence of the fibration above collapses. It implies that $\rho^*\colon H_T^*(X(h)) \to H^*(X(h))$ is surjective and induces a graded ring isomorphism

$$H^*(X(h)) \cong H_T^*(X(h))/(\pi^*(t_1),\dots,\pi^*(t_n)) \tag{2.4}$$

by (2.3). Therefore, one can find the ring structure on $H^*(X(h))$ through $H_T^*(X(h))$.

Since $H^{odd}(X(h)) = 0$, it follows from the localization theorem that the homomorphism

$$H_T^*(X(h)) \to H_T^*(X(h)^T) = \bigoplus_{w\in\mathfrak{S}_n} H_T^*(w) = \bigoplus_{w\in\mathfrak{S}_n} \mathbb{Z}[t_1,\dots,t_n] = \mathrm{Map}(\mathfrak{S}_n, \mathbb{Z}[t_1,\dots,t_n]) \tag{2.5}$$

induced from the inclusion map $X(h)^T \to X(h)$ is injective, where $X(h)^T$ is identified with $\mathfrak{S}_n$ by (2.2) and $\mathrm{Map}(P,Q)$ denotes the set of all maps from P to Q. The T-variety $X(h)$ is what is called a GKM manifold and the image of the homomorphism in (2.5) is described in [20] as follows;

$$\{f \in \mathrm{Map}(\mathfrak{S}_n, \mathbb{Z}[t_1,\dots,t_n]) \mid f(w) - f(w(i,j)) \in (t_{w(i)} - t_{w(j)}),\ \text{for } \forall w \in \mathfrak{S}_n,\ j < i \leq h(j)\}, \tag{2.6}$$

where (i,j) denotes the transposition interchanging i and j. We note that the image of $\pi^*(t_i) \in \pi^*(H^*(BT)) \subset H_T^*(X(h))$ by the homomorphism in (2.5) is the constant function $t_i \in \mathrm{Map}(\mathfrak{S}_n, \mathbb{Z}[t_1,\dots,t_n])$.

Guillemin and Zara [11] assigned a labeled graph to a GKM manifold. The labeled graph of $X(h)$ is as follows. The vertex set is the fixed point set $X(h)^T = \mathfrak{S}_n$. There is an edge between vertices w and v if and only if $v = w(i,j)$ for some $j \le i \le h(j)$, and the edge between w and $w(i,j)$ is labeled by $t_{w(i)} - t_{w(j)}$ up to sign.

Example 2.1. Let $n = 3$. For $h = (2,3,3)$ and $h' = (3,3,3)$, the labeled graphs of $X(h)$ and $X(h')$ are drawn in Figure 2, where we use the one-line notation for each vertex.

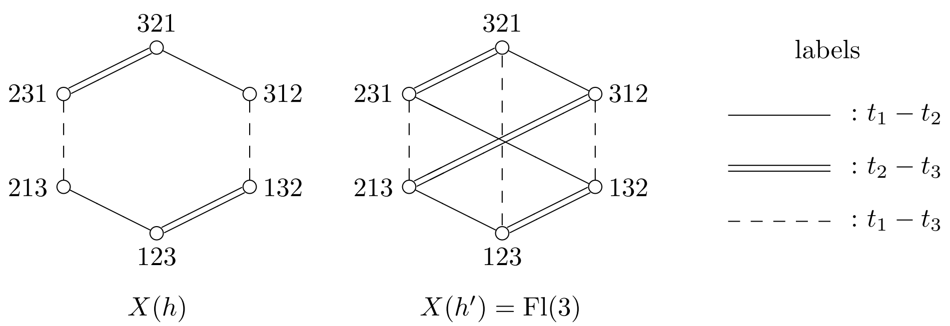

FIGURE 2. The labeled graphs of $X(h)$ and $X(h')$

In general, labeled graphs and their graph cohomologies are defined as follows.

Definition 2.2. Let R be a ring. A *labeled graph* (Γ, α) consists of a graph $\Gamma = (V,E)$ and a labeling $\alpha\colon E \to R$. The *graph cohomology* of a labeled graph (Γ,α) is defined as

$$H^*(\Gamma,\alpha) = \{f \in \mathrm{Map}(V,R) \mid f(w) - f(v) \in (\alpha(e)) \text{ for } \forall e = wv \in E\}.$$

The graph cohomology $H^*(\Gamma,\alpha)$ is a subring of $\mathrm{Map}(V,R)$ with the coordinate-wise sum and multiplication. Note that we may ignore the signs of the labels $\alpha(e)$ since $(\alpha(e)) = (-\alpha(e))$.

The observation above shows that the graph cohomology of the labeled graph of $X(h)$ coincides with $H_T^*(X(h))$.

Sending t_i to $t_{\sigma(i)}$ for $\sigma \in \mathfrak{S}_n$ and $i \in [n]$ induces an action of $\mathfrak{S}_n$ on $\mathbb{Z}[t_1,\dots,t_n]$. Then, the module $\mathrm{Map}(\mathfrak{S}_n, \mathbb{Z}[t_1,\dots,t_n])$ becomes an $\mathfrak{S}_n$-module under what is called the dot action defined by

$$(\sigma \cdot f)(w) := \sigma(f(\sigma^{-1}w)).$$

As easily checked, the graph cohomology of $X(h)$ is invariant under the dot action and $H_T^*(X(h))$ becomes a module over $\mathfrak{S}_n$. Moreover, since the action of $\mathfrak{S}_n$ preserves the ideal $(\pi^*(t_1),\dots,\pi^*(t_n))$, the action descends to $H^*(X(h))$ and $H^*(X(h))$ also becomes an module over $\mathfrak{S}_n$.

Obviously, constant functions in $\mathrm{Map}(\mathfrak{S}_n, \mathbb{Z}[t_1,\dots,t_n])$ satisfy the condition in (2.6). They are elements corresponding to $\pi^*(H^*(BT)) \subset H_T^*(X(h))$. Below are three types of elements x_i, $y_{j,k}$, and τ_A in $\mathrm{Map}(\mathfrak{S}_n, \mathbb{Z}[t_1,\dots,t_n])$ which satisfy the condition in (2.6), so they are in $H_T^*(X(h))$. Let

$$\begin{aligned} \perp(h) &:= \{j \in [n-1] \mid h(j-1) = h(j) = j+1\} \\ \mathsf{L}(h) &:= \{j \in [n-1] \mid h(j-1) = j \text{ and } h(j) = j+1\} \end{aligned} \tag{2.7}$$

where we understand $h(0) = 1$.

Definition 2.3. (1) For $i \in [n]$, $x_i(w) := t_{w(i)}$.

(2) For $j \in [n-1]$ with $j \in \perp(h)$ and $k \in [n]$,

$$y_{j,k}(w) := \begin{cases} t_k - t_{w(j+1)} & (\text{if } k \in \{w(1),\dots,w(j)\}) \\ 0 & (\text{otherwise}). \end{cases}$$

(3) For $A \subset [n]$ with $|A| \in \mathsf{L}(h)$

$$\tau_A(w) := \begin{cases} t_{w(|A|)} - t_{w(|A|+1)} & (\text{if } \{w(1), \dots, w(|A|)\} = A) \\ 0 & (\text{otherwise}). \end{cases}$$

The cohomological degrees of the elements $x_k, y_{j,k}, \tau_A$ are two. One can easily check that the dot actions of $\sigma \in \mathfrak{S}_n$ on these elements are given as follows:

$$\sigma \cdot x_k = x_k, \quad \sigma \cdot y_{j,k} = y_{j,\sigma(k)}, \quad \sigma \cdot \tau_A = \tau_{\sigma(A)}. \tag{2.8}$$

Remark 2.1. Here is a geometrical meaning of x_k's (regarded as elements in $H^2(X(h))$ through the isomorphism (2.4)). There is a nested sequence of tautological vector bundles over the flag variety $\mathrm{Fl}(n)$:

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n = \mathrm{Fl}(n) \times \mathbb{C}^n$$

where

$$\mathcal{F}_k := \{(V_\bullet, v) \in \mathrm{Fl}(n) \times \mathbb{C}^n \mid v \in V_k\} \qquad \text{and} \qquad V_\bullet = (\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n).$$

Then x_k ($k \in [n]$) is the image of the first Chern class of the quotient line bundle $\mathcal{F}_k/\mathcal{F}_{k-1}$ over $\mathrm{Fl}(n)$ by the homomorphism

$$\iota^* \colon H^*(\mathrm{Fl}(n)) \to H^*(X(h))$$

induced from the inclusion map $\iota\colon X(h) \to \mathrm{Fl}(n)$. The dot action on $H^*(\mathrm{Fl}(n))$ is trivial, so the image of ι^* must be contained in the ring of invariants $H^*(X(h))^{\mathfrak{S}_n}$. In fact, it follows from [1, Theorems A and B] that the image of ι^* agrees with $H^*(X(h))^{\mathfrak{S}_n}$ when tensoring with $\mathbb{Q}$ and

$$H^*(X(h))^{\mathfrak{S}_n} \otimes \mathbb{Q} = \mathbb{Q}[x_1, \dots, x_n]/(f_{h(1),1}, \dots, f_{h(n),n}) \tag{2.9}$$

where

$$f_{h(j),j} = \sum_{k=1}^{j} \left(x_k \prod_{\ell=j+1}^{h(j)} (x_k - x_\ell) \right). \tag{2.10}$$

In particular, the Hilbert series of $H^*(X(h))^{\mathfrak{S}_n}$ is given by

$$\mathrm{Hilb}(H^*(X(h))^{\mathfrak{S}_n}, \sqrt{q}) = \prod_{j=1}^{n-1} [h(j) - j]_q \tag{2.11}$$

where the Hilbert series of a graded algebra $\mathcal{A} = \sum_{r=0}^{\infty} \mathcal{A}^r$ over $\mathbb{Z}$ is defined as

$$\mathrm{Hilb}(\mathcal{A}, q) := \sum_{r=0}^{\infty} (\mathrm{rank}_{\mathbb{Z}} \mathcal{A}^r) q^r.$$

Through the isomorphism (2.4), the elements $x_k, y_{j,k}, \tau_A$ determine elements in $H^2(X(h))$, denoted by the same notation.

Theorem 2.4 ([4, Theorem 5.1])**.** *The elements*

$$\{x_k, y_{j,k}, \tau_A \mid k \in [n],\ j \in \perp(h) \backslash \{n-1\},\ A \subset [n] \text{ with } |A| \in \mathsf{L}(h) \backslash \{n-1\}\}$$

generate $H^2(X(h))$ with relations

(1) $\sum_{k=1}^n x_k = 0$,
(2) $\sum_{k=1}^n y_{j,k} = (x_1 + \cdots + x_j) - j x_{j+1}$ *for* $j \in \perp(h) \backslash \{n-1\}$,
(3) $\sum_{|A|=j} \tau_A = x_j - x_{j+1}$ *for* $j \in \mathsf{L}(h) \backslash \{n-1\}$.

Remark 2.2 (see Subsection 6.2 in [4] for more details)**.** The element $y_{j,k}$ is defined by looking at the j-th column of the configuration associated to the Hessenberg function h. Similarly, one can define an element $y^*_{i,k}$ of $H^*_T(\mathrm{Hess}(S, h))$ by looking at the i-th row of the configuration as follows. For $i \in [n]$, we define

$$h^*(i) := \min\{j \in [n] \mid h(j) \geq i\},$$

so that the shaded boxes in the i-th row and under the diagonal in the configuration associated to h are at positions (i,ℓ) $(h^*(i)\le \ell < i)$. When $h^*(i)=i-1$, we define

$$y^*_{i,k}(w) := \begin{cases} t_k - t_{w(i-1)} & (k\in\{w(i),\dots,w(n)\}) \\ 0 & (\text{otherwise}). \end{cases} \tag{2.12}$$

One can see that $y^*_{i.k}$ is in $H^2_T(\mathrm{Hess}(S,h))$ and we may replace $y_{j,k}$'s for $j\in\perp(h)\backslash\{n-1\}$ in the generating set in Theorem 2.4 by $y^*_{i,k}$'s for $i\ge 3$ such that $h^*(i)=h^*(i+1)=i-1$.

Example 2.2. When $h=(4,4,4,5,6,7,11,11,11,11,11)$ in Figure 1 (i.e. $(a,b)=(3,7)$), we have

$$\perp(h)=\{3,10\},\quad \mathsf{L}(h)=\{4,5,6\},$$

so Theorem 2.4 says that $H^2(X(h))$ is generated by

$$x_k\ (k\in[11]),\quad y_{3,k}\ (k\in[11]),\quad \tau_A \text{ for } A\subset[11] \text{ with } |A|=4,5 \text{ or } 6.$$

Moreover, it follows from Remark 2.2 that $y_{3,k}$ above may be replaced by $y^*_{8,k}$.

3. Necessity

In this section, we study a necessary condition on h for $H^*(X(h))$ to be generated in degree 2 as a ring.

3.1. Moment maps.

Let $\mu\colon \mathrm{Fl}(n)\to\mathbb{R}^n$ be the standard moment map on the flag variety $\mathrm{Fl}(n)$. Its image is the permutohedron Π_n obtained as the convex hull of the orbits of $(1,2,\dots,n)$ by permuting its coordinates. Indeed, if e_w $(w\in\mathfrak{S}_n)$ denotes the permutation flag associated with w, then we have

$$\mu(e_w)=(w^{-1}(1),\dots,w^{-1}(n))\in\mathbb{R}^n$$

(see [14, Lemma 3.1] for example). Let

$$\mathfrak{S}^r_n := \{w\in\mathfrak{S}_n \mid w(r)=n\}. \tag{3.1}$$

Then $\mu(\mathfrak{S}^r_n)$ is the set of all vertices of Π_n whose n-th coordinate is r. Therefore the projection

$$\pi_n\colon \Pi_n\to\mathbb{R},\qquad \pi_n(x_1,\dots,x_n)=x_n$$

on the n-th coordinate takes minimum on $\mathfrak{S}^1_n$ and maximum on $\mathfrak{S}^n_n$. The composition of μ and π_n

$$f:=\pi_n\circ\mu\colon \mathrm{Fl}(n)\to\mathbb{R} \tag{3.2}$$

is the moment map induced from the following S^1-action on $\mathbb{C}^n$

$$(z_1,\dots,z_n)\to(z_1,\dots,z_{n-1},gz_n)\quad (g\in S^1\subset\mathbb{C}), \tag{3.3}$$

and it is a Morse-Bott function.

Let h^j be the Hessenberg function obtained by removing all the boxes in the j-th row and all the boxes in the j-th column from its configuration (see Figure 3). To be precise, h^j is given as follows.

$$h^j(i)=\begin{cases} h(i) & (i<j,\ h(i)<j) \\ h(i)-1 & (i<j,\ h(i)\ge j) \\ h(i+1)-1 & (i\ge j) \end{cases}$$

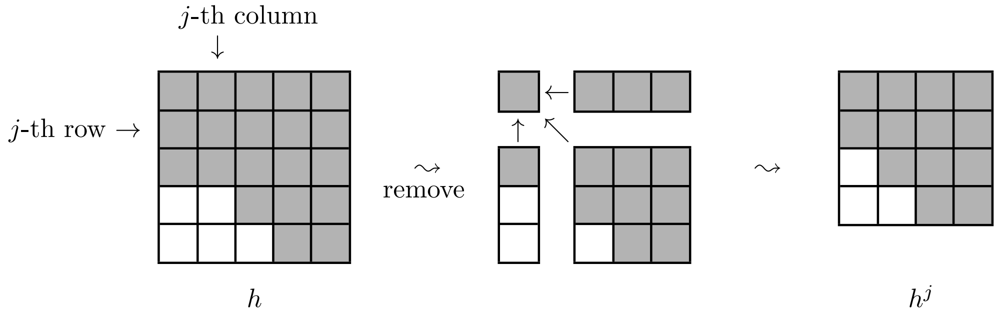

FIGURE 3. The configuration corresponding to h^j.

The following is a key lemma in our argument.

Lemma 3.1. *The restriction maps*

$$H^*(X(h);\mathbb{Q}) \to H^*(X(h^1);\mathbb{Q}), \quad H^*(X(h);\mathbb{Q}) \to H^*(X(h^n);\mathbb{Q})$$

are surjective.

Proof. Let f_h be the map f in (3.2) restricted to $X(h)$, which is also a Morse-Bott function. The inverse image of the minimum value under f_h is $X(h^1)$, so it follows from [19, Lemma 3.1] that the restriction map

$$H^*_{S^1}(X(h);\mathbb{Q}) \to H^*_{S^1}(X(h^1);\mathbb{Q}) \tag{3.4}$$

is surjective, where the S^1-action on $X(h)$ is the induced one from the S^1-action defined in (3.3). Since the S^1-action on $X(h^1)$ is trivial, we have $H^*_{S^1}(X(h^1);\mathbb{Q}) = H^*(BS^1;\mathbb{Q}) \otimes H^*(X(h^1);\mathbb{Q})$ and hence the forgetful map $H^*_{S^1}(X(h^1);\mathbb{Q}) \to H^*(X(h^1);\mathbb{Q})$ is surjective. Therefore, the surjectivity of (3.4) implies the surjectivity of the restriction map

$$H^*(X(h);\mathbb{Q}) \to H^*(X(h^1);\mathbb{Q})$$

in ordinary cohomology. The same argument applied to $-f_h$ proves the statement for $X(h^n)$. □

Remark 3.1. The surjectivity of the above restriction maps (even with $\mathbb{Z}$ coefficients) can also be verified by GKM theory as follows. Recall that the inclusion of the fixed point set induces an injective homomorphism $H^*_T(X(h)) \to H^*_T(X(h)^T) \cong \mathrm{Map}(\mathfrak{S}_n, H^*(BT))$. The equivariant cohomology $H^*_T(X(h))$ has an $H^*(BT)$-module basis $\{\sigma_{w,h} \mid w \in \mathfrak{S}_n\}$ (see [6, Definition 2.9 and Proposition 2.11]). It corresponds to a natural paving and then it is a 'flow-up basis.' Note that any element of $\mathfrak{S}^n_n = \mathfrak{S}_{n-1}$ is not greater than any element of $\mathfrak{S}_n \setminus \mathfrak{S}^n_n$. The restriction of $\{\sigma_{w,h} \mid w \in \mathfrak{S}^n_n\}$ onto $X(h^n)$, that is, its restriction onto the fixed point set $\mathfrak{S}^n_n = X(h^n)^T$ as elements of $\mathrm{Map}(\mathfrak{S}_n, H^*(BT))$, is a flow-up basis of $H^*_T(X(h^n))$. Hence $H^*_T(X(h)) \to H^*_T(X(h^n))$ is surjective, and then $H^*(X(h)) \to H^*(X(h^n))$ is also surjective. The surjectivity of $H^*(X(h)) \to H^*(X(h^1))$ can be verified by a similar argument.

Given a Hessenberg function h, we obtain a smaller Hessenberg function by removing the first column and row or the last column and row repeatedly, i.e. by taking h^1 or h^n repeatedly. We call it a minor of h. The following corollary follows from Lemma 3.1.

Corollary 3.2. *Let h' be a minor of h. If $H^*(X(h);\mathbb{Q})$ is generated in degree 2 as a ring, then so is $H^*(X(h');\mathbb{Q})$.*

An easy argument shows that h being of the form (1.1) can be rephrased as follows.

Proposition 3.3. *The Hessenberg function h is of the form* (1.1) *if and only if h has neither*

$$(\alpha,\beta,\dots,\beta),\ (\beta-1,\dots,\beta-1,\underbrace{\beta,\dots,\beta}_{\alpha})\ \text{for } 3 \le \alpha < \beta,\ \text{nor } (2,\gamma-1,\dots,\gamma-1,\gamma,\gamma)\ \text{for } \gamma \ge 5$$

as its minor.

Recall that if $h^\dagger$ denotes the Hessenberg function obtained by flipping the configuration of h along the anti-diagonal, then $X(h^\dagger) \cong X(h)$ as varieties. Therefore

$$X((\alpha, \beta, \dots, \beta)) \cong X((\beta-1, \dots, \beta-1, \underbrace{\beta, \dots, \beta}_{\alpha})).$$

Here, we know that $H^*(X((\alpha, \beta, \dots, \beta)); \mathbb{Q})$ is not generated in degree 2 for $3 \leq \alpha < \beta$ by [2, Theorem 4.3]. Thus, it suffices to treat the last case in Proposition 3.3, which we shall discuss in the next subsection.

3.2. **The case $h = (2, n-1, \dots, n-1, n, n)$.** In this subsection we prove the following proposition.

Proposition 3.4. *$H^*(X(h); \mathbb{Q})$ is not generated in degree 2 when $h = (2, n-1, \dots, n-1, n, n)$ for $n \geq 5$.*

Some computation is involved in the proof of this proposition but the idea of the proof is simple. We compute the Poincaré polynomial of $X(h)$ using Theorem 2.1(4). On the other hand, using explicit generators of $H^2(X(h))$ by [4], we compute an upper bound of the Hilbert series of the subring of $H^*(X(h))$ generated by $H^2(X(h))$. Then it turns out that the latter is strictly smaller than the former at a certain degree.

3.2.1. *Poincaré polynomial of $X(h)$.* The following proposition, which easily follows from Theorem 2.1(4), enables us to compute the Poincaré polynomial of $X(h)$ inductively.

Proposition 3.5 ([4, Proposition 3.1])**.**

$$\operatorname{Poin}(X(h), \sqrt{q}) = \sum_{j=1}^{n} q^{h(j)-j} \operatorname{Poin}(X(h^j), \sqrt{q}). \tag{3.5}$$

Using the proposition above, the Poincaré polynomial of $X(h)$ is explicitly computed as follows when $h = (h(1), n, \dots, n)$.

Proposition 3.6 ([2])**.** *When $h = (h(1), n \dots, n)$, we have*

$$\operatorname{Poin}(X(h), \sqrt{q}) = [h(1)]_q [n-1]_q! + (n-1) q^{h(1)-1} [n-h(1)]_q [n-2]_q!, \tag{3.6}$$

where

$$[m]_q = \frac{1-q^m}{1-q}, \quad [m]_q! = [1]_q [2]_q \cdots [m]_q = \prod_{j=1}^{m} \frac{1-q^j}{1-q}.$$

Now, let $h = (2, n-1, \dots, n-1, n, n)$ and set

$$P_n(q) := \operatorname{Poin}(X(h), \sqrt{q}).$$

Lemma 3.7. *For $n \geq 5$, the following recurrence formula holds*

$$\begin{aligned} P_n(q) = {} & (1+q)^2 [n-2]_q! + (n-2)(q+q^2)[n-3]_q [n-3]_q! \\ & + (n-1)(q+q^{n-3}) \{(1+q)[n-3]_q! + (n-3) q [n-4]_q [n-4]_q!\} \\ & + (q + q^2 + \cdots + q^{n-4}) P_{n-1}(q). \end{aligned}$$

Proof. Let $F_n(q)$ denote the right-hand side of (3.6) with $h(1) = 2$, that is,

$$F_n(q) := (1+q)[n-1]_q! + (n-1) q [n-2]_q [n-2]_q!. \tag{3.7}$$

Then we have

$$\begin{aligned} \operatorname{Poin}(X(h^1), \sqrt{q}) &= \operatorname{Poin}(X(h^n), \sqrt{q}) = F_{n-1}(q) \\ \operatorname{Poin}(X(h^2), \sqrt{q}) &= \operatorname{Poin}(X(h^{n-1}), \sqrt{q}) = (n-1) F_{n-2}(q) \\ \operatorname{Poin}(X(h^j), \sqrt{q}) &= P_{n-1}(q) \quad (3 \leq j \leq n-2), \end{aligned}$$

where we note that $X(h^2)$ consists of $n-1$ copies of $\mathrm{Fl}(n-2)$. Hence, by (3.5), we have

$$\begin{aligned}P_n(q) &= qF_{n-1}(q) + (n-1)q^{n-3}F_{n-2}(q) + (q^{n-4}+\cdots+q)P_{n-1}(q)\\ &\quad + (n-1)qF_{n-2}(q) + F_{n-1}(q)\\ &= (1+q)F_{n-1}(q) + (n-1)(q+q^{n-3})F_{n-2}(q) + (q+\cdots+q^{n-4})P_{n-1}(q).\end{aligned}$$

Combining this equation with (3.7), we obtain the desired equation. □

Lemma 3.8. *For $n \geq 4$, let*

$$Q_n(q) = (1+2nq+n(n-1)q^2)[n-2]_q! + \frac{n(n-3)}{2}q^{n-3}.$$

Then we have

$$P_n(q) \equiv Q_n(q) \mod (q^{n-2}).$$

In other words, $P_n(q)$ and $Q_n(q)$ coincide up to degree $n-3$.

Proof. We prove the lemma by induction on n. When $n=4$, we have

$$P_4(q) = 1+11q+11q^2+q^3, \quad Q_4(q) = 1+11q+20q^2+12q^3,$$

and the lemma is true for $n=4$.

Let n be given and suppose that the lemma is true for $n-1$, that is,

$$P_{n-1}(q) \equiv Q_{n-1}(q) \mod (q^{n-3}). \tag{3.8}$$

Hereafter, in this proof, all congruences will be taken modulo q^{n-2} unless otherwise stated. Since we have

$$\begin{aligned}(q+q^2)[n-3]_q[n-3]_q! &\equiv (q+q^2)[n-2]_q!\\ q^2[n-4]_q[n-4]_q! &\equiv q^2[n-3]_q!,\end{aligned}$$

the recurrence formula in Lemma 3.7 reduces to the following congruence relation:

$$\begin{aligned}P_n(q) &\equiv (1+nq+(n-1)q^2)[n-2]_q! + (n-1)(q+(n-2)q^2)[n-3]_q!\\ &\quad + (n-1)q^{n-3} + (q+\cdots+q^{n-4})P_{n-1}(q).\end{aligned} \tag{3.9}$$

It follows from (3.8) and the definition of Q_n that the sum of the last two terms above becomes as follows.

$$\begin{aligned}&(n-1)q^{n-3} + (q+\cdots+q^{n-4})P_{n-1}(q)\\ &\equiv (n-1)q^{n-3} + \left(1+(2n-2)q+(n-1)(n-2)q^2\right)(q+\cdots+q^{n-4})[n-3]_q! + \frac{(n-1)(n-4)}{2}q^{n-3}\\ &= \left\{1-q+(n-1)q(1-q)+nq+(n-1)^2q^2\right\}(q+\cdots+q^{n-4})[n-3]_q! + \frac{(n-1)(n-2)}{2}q^{n-3}\\ &\equiv \left\{q-q^{n-3}+(n-1)q^2+(nq+(n-1)^2q^2)(q+\cdots+q^{n-4})\right\}[n-3]_q! + \frac{(n-1)(n-2)}{2}q^{n-3}\\ &\equiv \left\{q+(n-1)q^2+(nq+(n-1)^2q^2)(q+\cdots+q^{n-4})\right\}[n-3]_q! + \frac{n(n-3)}{2}q^{n-3}\end{aligned}$$

By substituting it to (3.9), we obtain

$$\begin{aligned}P_n(q) &\equiv (1+nq+(n-1)q^2)[n-2]_q!\\ &\qquad + \left\{(nq+(n-1)^2q^2) + (nq+(n-1)^2q^2)(q+\cdots+q^{n-4})\right\}[n-3]_q! + \frac{n(n-3)}{2}q^{n-3}\\ &\equiv (1+nq+(n-1)q^2)[n-2]_q! + (nq+(n-1)^2q^2)[n-2]_q! + \frac{n(n-3)}{2}q^{n-3}\\ &= (1+2nq+n(n-1)q^2)[n-2]_q! + \frac{n(n-3)}{2}q^{n-3}\\ &= Q_n(q).\end{aligned}$$

This completes the induction step and the lemma has been proved. ☐

3.2.2. *Hilbert series of the subring generated by $H^2(X(h))$.* When $h = (2, n-1, \dots, n-1, n, n)$ for $n \geq 5$, we first observe $H^2(X(h))$. By (2.7), we have

$$\perp(h) = \{n-2\}, \quad \mathsf{L}(h) = \{1, n-1\}.$$

Therefore, it follows from Theorem 2.4 that $H^2(X(h))$ is generated by the following elements

$$x_k, \qquad y_k := y_{n-2,k}, \qquad \tau_k := \tau_{\{k\}} \qquad (k \in [n]), \tag{3.10}$$

where

$$\begin{aligned} &x_k(w) = t_{w(k)}, \\ &y_k(w) = y_{n-2,k}(w) = \begin{cases} t_k - t_{w(n-1)} & (\text{if } k \in \{w(1), \dots, w(n-2)\}) \\ 0 & (\text{otherwise}), \end{cases} \\ &\tau_k(w) = \tau_{\{k\}}(w) = \begin{cases} t_{w(1)} - t_{w(2)} & (\text{if } k = w(1)) \\ 0 & (\text{otherwise}) \end{cases} \end{aligned} \tag{3.11}$$

for $w \in \mathfrak{S}_n$ by Definition 2.3, and

$$\sum_{k=1}^{n} y_k = x_1 + \dots + x_{n-2} - (n-2)x_{n-1}, \qquad \sum_{k=1}^{n} \tau_k = x_1 - x_2 \tag{3.12}$$

by Theorem 2.4. We also have

$$\sigma \cdot x_k = x_k, \qquad \sigma \cdot y_k = y_{\sigma(k)}, \qquad \sigma \cdot \tau_k = \tau_{\sigma(k)}$$

for $\sigma \in \mathfrak{S}_n$ by (2.8).

To make the following argument clearer, we introduce elements ρ_k for $k \in [n]$ defined by

$$\rho_k(w) := \begin{cases} t_{w(n-1)} - t_{w(n)} & (\text{if } k = w(n)) \\ 0 & (\text{otherwise}). \end{cases} \tag{3.13}$$

Similarly to τ_k, the ρ_k satisfies the condition (2.6) so that it defines an element of $H^2_T(X(h))$ and $H^2(X(h))$ and

$$\sum_{k=1}^{n} \rho_k = x_{n-1} - x_n, \qquad \sigma \cdot \rho_k = \rho_{\sigma(k)} \quad \text{for } \sigma \in \mathfrak{S}_n. \tag{3.14}$$

An elementary check shows that

$$(y_k - y_\ell)(w) - (\rho_k - \rho_\ell)(w) = t_k - t_\ell \qquad (k, \ell \in [n],\ w \in \mathfrak{S}_n)$$

and hence $y_k - y_\ell = \rho_k - \rho_\ell$ in $H^2(X(h))$. Moreover, $\sum_{k=1}^{n} y_k$ and $\sum_{k=1}^{n} \rho_k$ are both linear polynomials in x_i's by (3.12) and (3.14), so we may replace y_k's in the generating set (3.10) by ρ_k's. Namely $H^2(X(h))$ is generated by

$$x_k, \qquad \tau_k, \qquad \rho_k \qquad (k \in [n])$$

with relations

$$\sum_{k=1}^{n} x_k = 0, \qquad \sum_{k=1}^{n} \tau_k = x_1 - x_2, \qquad \sum_{k=1}^{n} \rho_k = x_{n-1} - x_n, \tag{3.15}$$

and the actions of $\sigma \in \mathfrak{S}_n$ on those generators are given by

$$\sigma \cdot x_k = x_k, \qquad \sigma \cdot \tau_k = \tau_{\sigma(k)}, \qquad \sigma \cdot \rho_k = \rho_{\sigma(k)}. \tag{3.16}$$

Our purpose is to find a sharp upper bound of the Hilbert series of the subring $\mathcal{R}(h)$ of $H^*(X(h))$ generated by $H^2(X(h))$. Let $A(h)$ be the subring of $H^*(X(h))$ generated by x_k's and we regard $\mathcal{R}(h)$ as a module over $A(h)$. It follows from (3.11) and (3.13) that

$$\tau_k\tau_\ell = \begin{cases} (x_1-x_2)\tau_k & (k=\ell) \\ 0 & (k\neq\ell), \end{cases} \qquad \rho_k\rho_\ell = \begin{cases} (x_{n-1}-x_n)\rho_k & (k=\ell) \\ 0 & (k\neq\ell), \end{cases} \qquad \tau_k\rho_k = 0.$$

Therefore, $\mathcal{R}(h)$ is generated by 1, τ_k, ρ_k $(k\in[n])$, and $\tau_i\rho_j$ $(i\neq j\in[n])$ as a module over $A(h)$. The subring $A(h)$ itself is a submodule of $\mathcal{R}(h)$ over $A(h)$. We consider three other submodules of $\mathcal{R}(h)$ over $A(h)$:

$$\begin{aligned} B(h) :=& \{\sum_{k=1}^n b_k\tau_k \mid b_k\in A(h),\ \sum_{k=1}^n b_k = 0\}, \\ C(h) :=& \{\sum_{k=1}^n c_k\rho_k \mid c_k\in A(h),\ \sum_{k=1}^n c_k = 0\}, \\ D(h) :=& \{\sum_{1\le i,j\le n} d_{ij}\tau_i\rho_j \mid d_{ij}\in A(h),\ \sum_{j=1}^n d_{ij}=0 \text{ for } i\in[n],\ \sum_{i=1}^n d_{ij}=0 \text{ for } j\in[n]\} \end{aligned} \tag{3.17}$$

where $d_{kk}=0$ for $k\in[n]$. Note that $A(h)\otimes\mathbb{Q}$ agrees with the ring of invariants $H^*(X(h);\mathbb{Q})^{\mathfrak{S}_n}$ as mentioned in Remark 2.1.

Lemma 3.9. *$\mathcal{R}(h)$ is additively generated by $A(h), B(h), C(h)$, and $D(h)$ when tensoring with $\mathbb{Q}$.*

Proof. Since $H^*(X(h))$ is generated by 1, τ_k, ρ_k $(k\in[n])$, and $\tau_i\rho_j$ $(i\neq j\in[n])$ as a module over $A(h)$, it suffices to show that any element of the form

$$\sum_{k=1}^n b_k\tau_k + \sum_{k=1}^n c_k\rho_k + \sum_{1\le i,j\le n} d_{ij}\tau_i\rho_j \qquad (b_k,c_k,d_{ij}\in A(h),\ d_{kk}=0) \tag{3.18}$$

can be expressed as a sum of elements in $A(h)$, $B(h)$, $C(h)$, and $D(h)$ when tensoring with $\mathbb{Q}$.

Step 1. Set $b := \sum_{k=1}^n b_k$ and $c := \sum_{k=1}^n c_k$. Since $\sum_{k=1}^n \tau_k = x_1-x_2$ and $\sum_{k=1}^n \rho_k = x_{n-1}-x_n$ by (3.15), we have

$$\sum_{k=1}^n b_k\tau_k + \sum_{k=1}^n c_k\rho_k = \sum_{k=1}^n \left(b_k-\frac{b}{n}\right)\tau_k + \frac{b}{n}(x_1-x_2) + \sum_{k=1}^n\left(c_k-\frac{c}{n}\right)\rho_k + \frac{c}{n}(x_{n-1}-x_n).$$

Here the two sums at the right hand side above respectively belong to $B(h)\otimes\mathbb{Q}$ and $C(h)\otimes\mathbb{Q}$, and the remaining two terms belong to $A(h)\otimes\mathbb{Q}$.

Step 2. As for the last term in (3.18), since $\sum_{i=1}^n\tau_i = x_1-x_2$, we have

$$\begin{aligned} \sum_{1\le i,j\le n} d_{ij}\tau_i\rho_j &= \sum_{j=1}^n\left(\sum_{i=1}^n\left(d_{ij}-\frac{d_j}{n}\right)\tau_i\right)\rho_j + \sum_{j=1}^n\frac{d_j}{n}(x_1-x_2)\rho_j \\ &= \sum_{1\le i,j\le n}\tilde{d}_{ij}\tau_i\rho_j + \sum_{j=1}^n\frac{d_j}{n}(x_1-x_2)\rho_j \end{aligned} \tag{3.19}$$

where

$$d_j := \sum_{i=1}^n d_{ij} \quad\text{and}\quad \tilde{d}_{ij} := d_{ij}-\frac{d_j}{n}.$$

The last sum in (3.19) is a sum of elements in $A(h)\otimes\mathbb{Q}$ and $C(h)\otimes\mathbb{Q}$ by Step 1. We shall show that the sum $\sum_{1\le i,j\le n}\tilde{d}_{ij}\tau_i\rho_j$ in (3.19) is a sum of elements in $A(h)\otimes\mathbb{Q}$, $B(h)\otimes\mathbb{Q}$, and $D(h)\otimes\mathbb{Q}$. We note that

$$\sum_{i=1}^n\tilde{d}_{ij} = \sum_{i=1}^n\left(d_{ij}-\frac{d_j}{n}\right) = \sum_{i=1}^n d_{ij} - d_j = 0 \tag{3.20}$$

and set

$$\tilde{d}_i := \sum_{j=1}^{n} \tilde{d}_{ij}. \tag{3.21}$$

Since $\sum_{j=1}^{n} \rho_j = x_{n-1} - x_n$, we have

$$\sum_{1\le i,j\le n} \tilde{d}_{ij}\tau_i\rho_j = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\tilde{d}_{ij} - \frac{\tilde{d}_i}{n}\right)\rho_j\right)\tau_i + \sum_{i=1}^{n}\frac{\tilde{d}_i}{n}(x_{n-1}-x_n)\tau_i. \tag{3.22}$$

Here the second sum at the right hand side of (3.22) is a sum of elements in $A(h)\otimes\mathbb{Q}$ and $B(h)\otimes\mathbb{Q}$ by Step 1. As for the coefficients $\tilde{d}_{ij} - \frac{\tilde{d}_i}{n}$ of $\tau_i\rho_j$ in the first sum at the right hand side of (3.22), it follows from (3.20) and (3.21) that we have

$$\sum_{i=1}^{n}\left(\tilde{d}_{ij} - \frac{\tilde{d}_i}{n}\right) = \sum_{i=1}^{n}\tilde{d}_{ij} - \frac{1}{n}\sum_{i=1}^{n}\tilde{d}_i = -\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{d}_{ij} = -\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\tilde{d}_{ij}\right) = 0,$$
$$\sum_{j=1}^{n}\left(\tilde{d}_{ij} - \frac{\tilde{d}_i}{n}\right) = \sum_{j=1}^{n}\tilde{d}_{ij} - \tilde{d}_i = 0.$$

Thus, the first sum at the right hand side of (3.22) belongs to $D(h)\otimes\mathbb{Q}$. This completes the proof of the lemma. □

We shall calculate upper bounds of the Hilbert series of $A(h), B(h), C(h)$, and $D(h)$.

Hilbert series of $A(h)$. Since $A(h)\otimes\mathbb{Q} = H^*(X(h))^{\mathfrak{S}_n}\otimes\mathbb{Q}$ and $h=(2,n-1,\dots,n-1,n,n)$ in our case, it follows from (2.11) that

$$\mathrm{Hilb}(A(h),\sqrt{q}) = \prod_{j=1}^{n-1}[h(j)-j]_q = (1+q)^2[n-2]_q!. \tag{3.23}$$

Hilbert series of $B(h)$. It follows from(3.11) that $(x_1-t_k)\tau_k$ vanishes at every $w\in\mathfrak{S}_n$, so we have

$$(x_1-t_k)\tau_k = 0 \quad \text{in } H_T^*(X(h)) \quad \text{and hence} \quad x_1\tau_k = 0 \quad \text{in } H^*(X(h)). \tag{3.24}$$

Therefore, $B(h)$ is indeed a module over $A(h)/(x_1)$. Here

$$A(h)/(x_1)\otimes\mathbb{Q} = A(h^1)\otimes\mathbb{Q}$$

by (2.9) and (2.10). Since $h^1 = (n-2,\dots,n-2,n-1,n-1)$, it follows from (2.11) that

$$\mathrm{Hilb}(A(h)/(x_1),\sqrt{q}) = \prod_{j=1}^{n-2}[h^1(j)-j]_q = (1+q)[n-2]_q!.$$

Since $B(h)$ is a module over $A(h)/(x_1)$ generated by $\tau_i - \tau_{i+1}$ $(i\in[n-1])$ and the cohomological degrees of τ_k's are two, we obtain an upper bound of $\mathrm{Hilb}(B(h),q)$ as follows:

$$\mathrm{Hilb}(B(h),\sqrt{q}) \le (n-1)q\,\mathrm{Hilb}(A(h)/(x_1),\sqrt{q}) = (n-1)(q+q^2)[n-2]_q!. \tag{3.25}$$

Here $\sum_{i=0}^{\infty} a_iq^i \le \sum_{i=0}^{\infty} b_iq^i$ $(a_i,b_i\in\mathbb{Z})$ means that $a_i\le b_i$ for all i's.

Hilbert series of $C(h)$. To $f\in\mathrm{Map}(\mathfrak{S}_n,\mathbb{Z}[t_1,\dots,t_n])$ we associate $f^\vee\in\mathrm{Map}(\mathfrak{S}_n,\mathbb{Z}[t_1,\dots,t_n])$ defined by

$$f^\vee(w) := f(ww_0) \qquad \text{for } w\in\mathfrak{S}_n,$$

where w_0 denotes the longest element in $\mathfrak{S}_n$, i.e. $w_0 = n\ n-1\ \cdots\ 2\ 1$ in one-line notation. This defines an involution on $\mathrm{Map}(\mathfrak{S}_n,\mathbb{Z}[t_1,\dots,t_n])$ and one can easily check that

$$x_k^\vee = x_{n-k+1}, \qquad \tau_k^\vee = -\rho_k, \qquad \rho_k^\vee = -\tau_k$$

from (3.11) and (3.13). Hence the involution gives an isomorphism between $B(h)$ and $C(h)$, and the same inequality as (3.25) holds for $C(h)$, i.e.

$$\mathrm{Hilb}(C(h),\sqrt{q}) \le (n-1)(q+q^2)[n-2]_q!. \tag{3.26}$$

Hilbert series of $D(h)$. We have $x_1\tau_k = 0$ by (3.24). Similarly we have $x_n\rho_k = 0$ since $(x_1\tau_k)^\vee = -x_n\rho_k$. (The fact $x_n\rho_k = 0$ also follows from the definition (3.11) and (3.13) of x_k and ρ_k.) Therefore, $D(h)$ is indeed a module over $A(h)/(x_1, x_n)$.

As mentioned in Remark 2.1, $A(h)\otimes\mathbb{Q} = H^*(X(h))^{\mathfrak{S}_n}\otimes\mathbb{Q}$ and it is the image of the restriction map $\iota^*\colon H^*(\mathrm{Fl}(n)) \to H^*(X(h))$. Therefore, $A(h)/(x_1,x_n)$ is the image of the restriction map from $H^*(\mathrm{Fl}(n-2))$ and hence

$$\mathrm{Hilb}(A(h)/(x_1,x_n),\sqrt{q}) \le [n-2]_q!.$$

(In fact, the equality holds above.) There are $2n$ relations among d_{ij} $(i\neq j)$ in the definition (3.17) of $D(h)$, but one relation can be obtained from the other $2n-1$ relations because $\sum_{i=1}^n\left(\sum_{j=1}^n d_{ij}\right) = \sum_{j=1}^n\left(\sum_{i=1}^n d_{ij}\right)$. Moreover, there are $n(n-1)$ number of d_{ij}'s and the cohomological degree of $\tau_i\rho_j$ is four. Thus

$$\mathrm{Hilb}(D(h),\sqrt{q}) \le \mathrm{Hilb}(A(h)/(x_1,x_n),\sqrt{q})\,\{n(n-1)-(2n-1)\}\,q^2 \le (n^2-3n+1)q^2[n-2]_q!. \tag{3.27}$$

Proof of Proposition 3.4. It follows from Lemma 3.9, (3.23), (3.25), (3.26), and (3.27) that

$$\begin{aligned}\mathrm{Hilb}(\mathcal{R}(h),\sqrt{q}) &\le (1+q)^2[n-2]_q! + 2(n-1)(q+q^2)[n-2]_q! + (n^2-3n+1)q^2[n-2]_q!\\ &= (1+2nq+n(n-1)q^2)[n-2]_q!.\end{aligned}$$

The coefficient of q^{n-3} in the last term above is less than that of $P_n(q)$ in Lemma 3.8 by $n(n-3)/2$, proving the proposition. □

4. Sufficiency

The purpose of this section is devoted to the proof of the sufficiency of Theorem 1.1. Indeed, using an idea in [15], we will show that when the Hessenberg function h is of the form (1.1), $X(h)$ is a fiber bundle over a compact smooth toric variety with a product of flag varieties as the fiber. This implies that the cohomology ring of $X(h)$ is generated in degree two because so are the cohomology rings of the base space and the fiber.

Let $a,b\in[n]$ with $a<b$. W denote by $\mathrm{Fl}_{[a,b]}(n)$ the partial flag variety consisting of a sequence of linear subspaces in $\mathbb{C}^n$ with consecutive dimensions $a, a+1, \dots, b$. We also denote $\mathrm{Fl}(\mathbb{C}^n)$ simply by $\mathrm{Fl}(n)$. We consider a map

$$\pi_{[a,b]}\colon \mathrm{Fl}(n)\to \mathrm{Fl}_{[a,b]}(n) \tag{4.1}$$

defined by

$$\pi_{[a,b]}(V_1\subset V_2\subset\cdots\subset V_n) := (V_a\subset V_{a+1}\subset\cdots\subset V_b).$$

The inverse image of the partial flag $(V_a\subset V_{a+1}\subset\cdots\subset V_b)$ by $\pi_{[a,b]}$ is identified with the set of pairs of partial flags in $\mathbb{C}^n$:

$$F_{[a,b]} := \{((V_1\subset\cdots\subset V_a),(V_b\subset\cdots\subset V_n))\}.$$

Since V_a and V_b are fixed, the former elements above form a complete flag variety in V_a, that is isomorphic to $\mathrm{Fl}(a)$. Taking quotients by V_b for the latter flags above, one sees that they form a variety isomorphic to $\mathrm{Fl}(n-b)$. Therefore, the map $\pi_{[a,b]}$ in (4.1) provides a fibration with the fiber $F_{[a,b]}$ isomorphic to $\mathrm{Fl}(a)\times\mathrm{Fl}(n-b)$:

$$F_{[a,b]}\to\mathrm{Fl}(n)\xrightarrow{\pi_{[a,b]}}\mathrm{Fl}_{[a,b]}(n).$$

For our h, $V_\bullet = (V_1\subset V_2\subset\cdots\subset V_n)\in\mathrm{Fl}(n)$ is in $X(h)$ if and only if

$$SV_k\subset\begin{cases}V_{a+1} & (k\le a)\\ V_{k+1} & (a\le k\le b-1)\\ V_n=\mathbb{C}^n & (b\le k\le n)\end{cases}$$

Therefore, if $V_\bullet$ is in $X(h)$, then $\pi_{[a,b]}(V_\bullet) = (V_a \subset V_{a+1} \subset \cdots \subset V_b)$ satisfies the condition

$$SV_k \subset V_{k+1} \quad (a \le \forall k \le b-1). \tag{4.2}$$

Conversely, if a partial flag $(V_a \subset V_{a+1} \subset \cdots \subset V_b)$ satisfies the condition (4.2), then any complete flag $V_\bullet \in \mathrm{Fl}(n)$ extending this partial flag is in $X(h)$. Indeed, $SV_k \subset V_{a+1}$ for $k \le a$ is satisfied for any choice of V_k because $V_k \subset V_a$ for $k \le a$ and $SV_a \subset V_{a+1}$. Moreover, $SV_k \subset V_n = \mathbb{C}^n$ for $b \le k \le n$ is trivially satisfied for any choice of V_k. Therefore, if we set

$$Y_{[a,b]} := \{(V_a \subset \cdots \subset V_b) \in \mathrm{Fl}_{[a,b]}(n) \mid SV_k \subset V_{k+1} \quad (a \le \forall k \le b-1)\}$$

then $\pi_{[a,b]}(X(h)) = Y_{[a,b]}$ and $\pi_{[a,b]}$ restricted to $X(h)$, also denoted by $\pi_{[a,b]}$, provides a fibration with fiber $F_{[a,b]}$:

$$F_{[a,b]} \to X(h) \xrightarrow{\pi_{[a,b]}} Y_{[a,b]}.$$

Lemma 4.1. *$Y_{[a,b]}$ is a compact smooth toric variety of dimension $n-1$.*

Proof. Since $\pi_{[a,b]} \colon X(h) \to Y_{[a,b]}$ is a fibration and $X(h)$ is a compact smooth variety, $Y_{[a,b]}$ is also a compact smooth variety. Moreover, since the fiber $F_{[a,b]}$ is isomorphic to $\mathrm{Fl}(a) \times \mathrm{Fl}(n-b)$, it follows from Theorem 2.1(2) that

$$\begin{aligned}\dim Y_{[a,b]} &= \dim X(h) - \dim F_{[a,b]} \\ &= \Big(\frac{1}{2}a(a+1) + b - a - 1 + \frac{1}{2}(n-b)(n-b+1)\Big) - \Big(\frac{1}{2}a(a-1) + \frac{1}{2}(n-b)(n-b-1)\Big) \\ &= n-1.\end{aligned}$$

We shall prove that the action of $(\mathbb{C}^*)^n$ on $Y_{[a,b]}$ has an orbit of dimension $n-1$, which implies that $Y_{[a,b]}$ is a toric variety because it is a smooth variety of dimension $n-1$. We take our semisimple matrix S to be a diagonal matrix, so that the diagonal entries denoted by $s_1, \dots, s_n$ are mutually distinct. Let $\mathbf{g} = {}^t(g_1, \dots, g_n) \in (\mathbb{C}^*)^n$. Then vectors $\mathbf{g}, S\mathbf{g}, \dots, S^{k-1}\mathbf{g}$ for $1 \le k \le n$ are linearly independent because $s_1, \dots, s_n$ are mutually distinct. Therefore, the linear subspace $V_k(\mathbf{g})$ spanned by those vectors are of dimension k and $SV_k(\mathbf{g}) \subset V_{k+1}(\mathbf{g})$ for $1 \le k \le n-1$. Hence, we have

$$V(\mathbf{g}) := \Big(V_a(\mathbf{g}) \subset V_{a+1}(\mathbf{g}) \subset \cdots \subset V_b(\mathbf{g})\Big) \in Y_{[a,b]} \quad \text{for any } \mathbf{g} \in (\mathbb{C}^*)^n.$$

Let $\mathbf{1} := {}^t(1, \dots, 1)$. Then $V_k(\mathbf{g}) = \mathbf{g}V_k(\mathbf{1})$ for any k and hence $V(\mathbf{g}) = \mathbf{g}V(\mathbf{1})$. This shows that the set $\{V(\mathbf{g}) \mid \mathbf{g} \in (\mathbb{C}^*)^n\}$ is the $(\mathbb{C}^*)^n$-orbit of $V(\mathbf{1})$. If $g_p = g_q$ for any p and q, then it is obvious that $\mathbf{g}V(\mathbf{1}) = V(\mathbf{1})$. The converse is also true. In fact, it is true that if $\mathbf{g}V_k(\mathbf{1}) = V_k(\mathbf{1})$ for some $k \le n-1$, then $g_p = g_q$ for any p and q. The proof is as follows. For $k \le n-1$, we set

$$\Lambda_k := \{I \mid I \subset [n],\ |I| = k\}.$$

For $I \in \Lambda_k$, we denote by $d_I(\mathbf{g})$ the determinant of the submatrix formed by all i-th rows in $[\mathbf{g}, S\mathbf{g}, \dots, S^{k-1}\mathbf{g}]$ for $i \in I$. Then

$$[d_I(\mathbf{g})]_{I \in \Lambda_k} \in \mathbb{C}P^{\binom{n}{k}-1}$$

is the Plücker coordinate of $V_k(\mathbf{g})$. Since $d_I(\mathbf{g}) = \mathbf{g}_I d_I(\mathbf{1})$, where $\mathbf{g}_I := \prod_{i \in I} g_i$, we have $[d_I(\mathbf{g})]_{I \in \Lambda_k} = [\mathbf{g}_I d_I(\mathbf{1})]_{I \in \Lambda_k}$. Here, $d_I(\mathbf{1}) \neq 0$ for any $I \in \Lambda_k$ because $d_I(\mathbf{1})$ is the Vandermonde's determinant of s_i's $(i \in I)$ and all s_i's are mutually distinct. It follows that $[d_I(\mathbf{g})]_{I \in \Lambda_k} = [d_I(\mathbf{1})]_{I \in \Lambda_k}$ if and only if $\mathbf{g}_I = \mathbf{g}_J$ for any $I, J \in \Lambda_k$, and this means that $g_p = g_q$ for any p and q since $k \le n-1$. Therefore, the $(\mathbb{C}^*)^n$-orbit of $V(\mathbf{1})$ is of dimension $n-1$. □

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